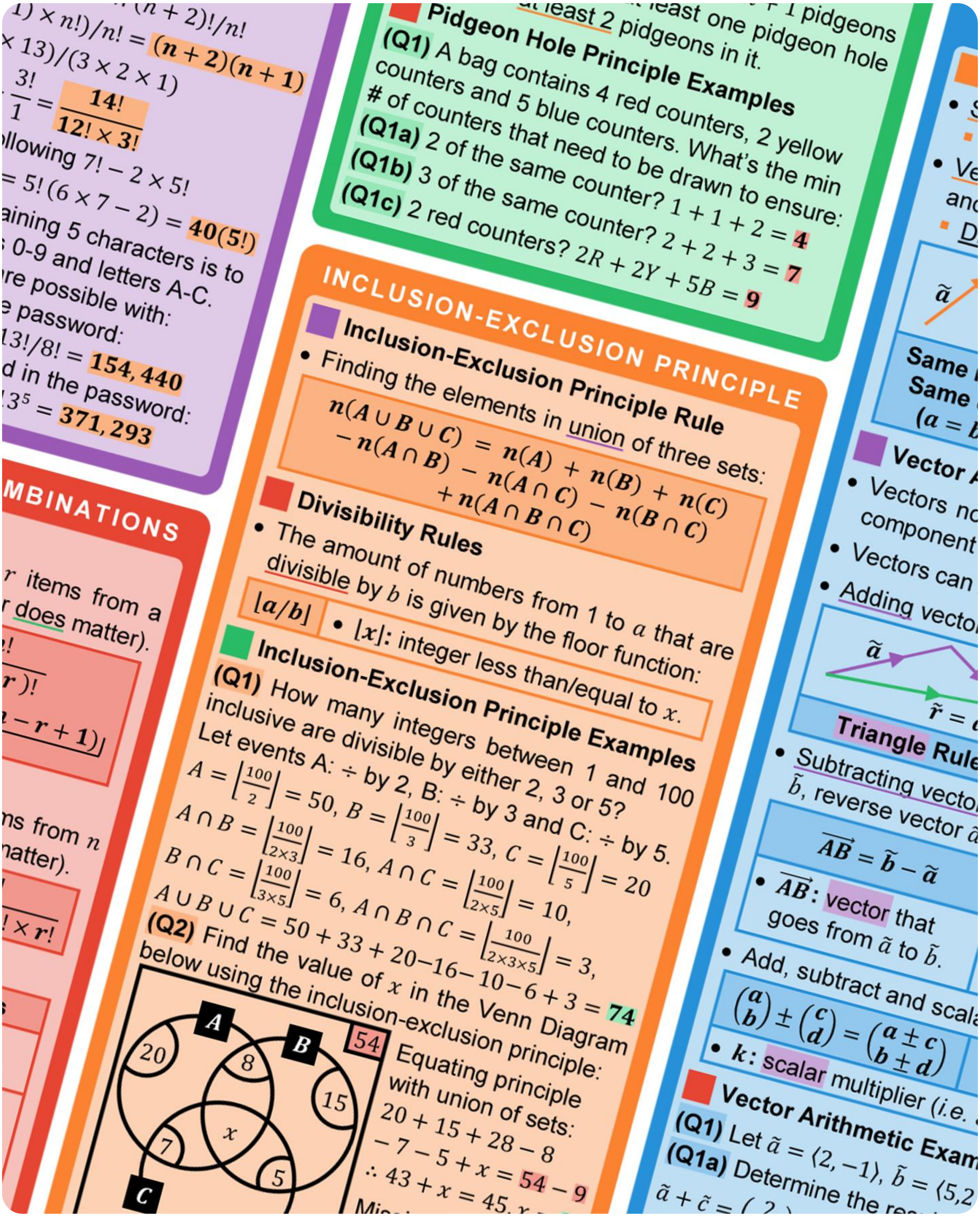

ATAR Mathematics Specialist Units 1 & 2

Exam Notes for Western Australian Year 11 Students

Created by Anthony Bochrinis
Version 3.0 (Updated 21/12/19)

REVISE ONLINE

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▶ About the Author - Anthony Bochrinis

Hello! My name is Anthony and I graduated from high school in 2012, completed a Bachelor of Actuarial Science in 2015, completed my Graduate Diploma in Secondary Education in 2017 and am now a secondary mathematics teacher!

My original exam notes (created in 2013) were inspired by Severus Snape's copy of Advanced Potion Making in Harry Potter and the Half-Blood Prince; a textbook filled with annotations containing all of the pro tips and secrets to help gain a clearer understanding.

Thank you for being a part of my journey in realising that teaching is my lifelong vocation!

COMBINATORICS

PRINCIPLES OF COUNTING

Multiplication Principle

- If there are A choices in one event and B choices in another event, then total choices:

$n(Event) = n(A) \times n(B)$

Addition Principle

- If there are A and B disjoint events (i.e. they have no common outcomes) then the total number of outcomes for the event is:

$n(Event) = n(A) + n(B)$

ARRANGEMENTS

Factorials ($n!$)

- The product of all positive integers less than or equal to a number n (e.g. $3! = 3 \times 2 \times 1$).

$n!$ pronounced **"n factorial"**, for $n > 0$:
$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

- Exception to calculating factorials:

Factorial rule exception: $0! = 1$
As there is 1 way to arrange 0 objects

Line, Repeated & Circular Arrangements

- The number of ways to arrange n distinct objects in a line is given by the equation:

Line Arrangement: $n!$

- The number of ways of arranging n objects of which a of one type are alike, b of a second type are alike, c of a third type are alike etc:
  - Find how many ways to arrange the word STATISTICS $= 10!/(3!\,2!\,3!) = 50{,}400$

Repeated Arrangement: $\dfrac{n!}{a! \times b! \times c! \times \ldots}$

- The number of ways to arrange n distinct objects in a circle is given by the equation:

Circular Arrangement: $(n-1)!$

If clockwise and anti-clockwise arrangements are the same: $0.5(n-1)!$

Cluster and Complement Arrangements

- Clustering refers to arrangements that require two or more objects to be together:
  - E.g. Find the number of ways to arrange the letters in WORDS given W and O must be together: $(5-2+1)! \times 2! = 48$

Cluster: $(n-r+1)! \times r!$

- Complement refers to arrangements that restricts two objects from being together:
  - E.g. Find the number of ways to arrange the letters in WORDS given W and O must not be together: $5! - 2!\,4! = 72$

Complement: $n! - 2!\,(n-1)!$ or
$n(complement) = n(total) - n(together)$

Arrangement/Factorial Examples

(Q1) Simplify the expression $(n+2)!/n!$
$= ((n+2) \times (n+1) \times n!)/n! = (n+2)(n+1)$

(Q2) Simplify $(14 \times 13)/(3 \times 2 \times 1)$
$= \dfrac{14 \times 13}{3 \times 2 \times 1} = \dfrac{14!}{12!} \div \dfrac{3!}{1} = \dfrac{14!}{12! \times 3!}$

(Q3) Factorise the following $7! - 2 \times 5!$
$= 7 \times 6 \times 5! - 2 \times 5! = 5!\,(6 \times 7 - 2) = 40(5!)$

(Q4) A password containing 5 characters is to be made from numbers 0-9 and letters A-C. How many passwords are possible with:

(Q4a) No repetition in the password:
$13 \times 12 \times 11 \times 10 \times 9 = 13!/8! = 154{,}440$

(Q4b) Repetition is allowed in the password:
$13 \times 13 \times 13 \times 13 \times 13 = 13^5 = 371{,}293$

PERMUTATIONS / COMBINATIONS

Permutations (Listing)

- Number of ways of picking r items from a collection of n items (i.e. order does matter).

$${}^{n}P_{r} = n\ pick\ r = \frac{n!}{(n-r)!} = \underbrace{n \times (n-1) \times (n-2) \ldots \times (n-r+1)}_{r\ terms}$$

Combinations (Grouping)

- Number of ways of choosing r items from n possible items (i.e. order does not matter).

$${}^{n}C_{r} = n\ choose\ r = \binom{n}{r} = \frac{n!}{(n-r)! \times r!}$$

Permutations vs. Combinations

Permutations	Combinations
Order does matter (picking)	Order doesn't matter (grouping)
Picking a principal and vice principal from 10 candidates.	Picking two school leaders from 10 candidates

▶ Topic Is Continued In Next Column ◀

PERMUTATIONS / COMBINATIONS

Common Counting Techniques

- Adding together all consecutive numbers:

$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$

Permutations/Combinations Examples

(Q1) From 7 boys and 6 girls, 5 are selected to form a team so that at least 4 boys are on the team. How many teams are possible?
$= \left[\binom{7}{4} \times \binom{6}{1}\right] + \binom{7}{5} = 35 \times 6 + 21 = 231$

(Q2) From 20 candidates (11 males and 9 females), 5 are chosen to form a council. How many combinations of different councils can be formed if there needs to be at least 3 males?
$= (3M\ and\ 2F) + (4M\ and\ 1F) + (5M\ and\ 1F)$
$= \left[\binom{11}{3} \times \binom{9}{2}\right] + \left[\binom{11}{4} \times \binom{9}{1}\right] + \left[\binom{11}{5} \times \binom{9}{0}\right]$
$= 5{,}940 + 2{,}970 + 462 = 9{,}372\ councils$

(Q3) Find the probability of rolling three dice, that are labelled 1-9 inclusive, having a sum that equals an odd number?
3 odd or 1 odd / 2 even produce an odd sum.
Out of numbers 1-9 inclusive, 5 odd & 4 even.
$= \left[\dfrac{\binom{5}{3} \times \binom{4}{0}}{\binom{9}{3}}\right] + \left[\dfrac{\binom{5}{1} \times \binom{4}{2}}{\binom{9}{3}}\right] = \dfrac{10}{21} = 0.4762$

(Q4) 3 girls and 4 boys are chosen from 7 girls and 6 boys to sit in a row where the boys have to sit next to each other. How many possible ways are there for the 7 students to sit?
Choosing students: $\binom{7}{3} \times \binom{6}{4} = 525$
Arranging students: $(7-4+1)!\,4! = 576$
Total combinations: $525 \times 576 = 302{,}400$

(Q5) The digits from 0-7 inclusive are used without repetition to make 4-digit numbers that cannot start with the number 0.

(Q5a) How many 4-digit numbers are even?
Digit "0" is last: $7 \times 6 \times 5 \times 1 = 210$
Digit "2" is last: $6 \times 6 \times 5 \times 1 = 180$
Digit "4" is last: $6 \times 6 \times 5 \times 1 = 180$
Digit "6" is last: $6 \times 6 \times 5 \times 1 = 180$
$210 + 3 \times 180 = 750$

(Q5b) How many 4-digit even numbers >4000?
Digit "0" is last: $4 \times 6 \times 5 \times 1 = 120$
Digit "2" is last: $4 \times 6 \times 5 \times 1 = 120$
Digit "4" is last: $3 \times 6 \times 5 \times 1 = 90$
Digit "6" is last: $3 \times 6 \times 5 \times 1 = 90$
$2 \times 120 + 2 \times 190 = 420$

(Q6) Four-character passwords can be made from digits 0-5 inclusive and the alphabet without any characters repeating. How many possible passwords start with a letter?
$= 26 \times 31 \times 30 \times 29 = 26 \times {}^{31}P_3 = 701{,}220$

PIDGEON HOLE PRINCIPLE

Pidgeon Hole Principle and Rules

- If there's n pidgeon holes and $n+1$ pidgeons go into them then at least one pidgeon hole will have at least 2 pidgeons in it.

Pidgeon Hole Principle Examples

(Q1) A bag contains 4 red counters, 2 yellow counters and 5 blue counters. What's the min # of counters that need to be drawn to ensure:
(Q1a) 2 of the same counter? $1 + 1 + 2 = 4$
(Q1b) 3 of the same counter? $2 + 2 + 3 = 7$
(Q1c) 2 red counters? $2R + 2Y + 5B = 9$

INCLUSION-EXCLUSION PRINCIPLE

Inclusion-Exclusion Principle Rule

- Finding the elements in union of three sets:

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

Divisibility Rules

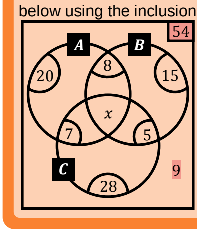
- The amount of numbers from 1 to a that are divisible by b is given by the floor function:

$\lfloor a/b \rfloor$ • $\lfloor x \rfloor$: integer less than/equal to x.

Inclusion-Exclusion Principle Examples

(Q1) How many integers between 1 and 100 inclusive are divisible by either 2, 3 or 5?
Let events A: ÷ by 2, B: ÷ by 3 and C: ÷ by 5.
$A = \left\lfloor \frac{100}{2} \right\rfloor = 50$, $B = \left\lfloor \frac{100}{3} \right\rfloor = 33$, $C = \left\lfloor \frac{100}{5} \right\rfloor = 20$
$A \cap B = \left\lfloor \frac{100}{2\times3} \right\rfloor = 16$, $A \cap C = \left\lfloor \frac{100}{2\times5} \right\rfloor = 10$,
$B \cap C = \left\lfloor \frac{100}{3\times5} \right\rfloor = 6$, $A \cap B \cap C = \left\lfloor \frac{100}{2\times3\times5} \right\rfloor = 3$,
$A \cup B \cup C = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$

(Q2) Find the value of x in the Venn Diagram below using the inclusion-exclusion principle:



Equating principle with union of sets:
$20 + 15 + 28 - 8 - 7 - 5 + x = 54 - 9$
$\therefore 43 + x = 45, x = 2$
Missing regions:
$A \cap B = 8 - 2 = 6$
$A \cap C = 7 - 2 = 5$
$B \cap C = 3 - 2 = 3$

PASCAL'S TRIANGLE

Pascal's Triangle

- First seven rows:

Row 0	1						
Row 1	1	1					
Row 2	1	2	1				
Row 3	1	3	3	1			
Row 4	1	4	6	4	1		
Row 5	1	5	10	10	5	1	
…	1	6	15	20	15	6	1

Column 0, Column 1, Column 2, Column 3, Column 4, Col. 5 …

Combinations and Pascal's Triangle

- Creating Pascal's Triangle laws:

${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

${}^{n}C_r = \dfrac{n}{r} \times {}^{n-1}C_{r-1}$

- Finding combinations in Pascal's Triangle:

$\binom{n}{r} = \left(\begin{matrix} Row\ \#of\ Pascal's\ Tri. \\ Column\ \#of\ Pascal's\ Tri. \end{matrix}\right)$

- Pascal's Triangle vertical symmetry laws:

${}^{n}C_r = {}^{n}C_{n-r}$

${}^{n}C_1 = {}^{n}C_{n-1} = n$ ${}^{n}C_0 = {}^{n}C_n = 1$

- Sum of all elements in each row:

${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \cdots + {}^{n}C_n = 2^n$

Expanding Brackets with Two Terms

- Expanding binomials to large powers of n:

Expanding $(ax+by)^n$
$= \binom{n}{0}(ax)^n(by)^0 + \binom{n}{1}(ax)^{n-1}(by)^1 + \cdots + \binom{n}{n-1}(ax)^1(by)^{n-1} + \binom{n}{n}(ax)^0(by)^n$

Pascal's Triangle Examples

(Q1) From a group of 10 teachers, how many ways can at least 2 teachers be selected?
$\binom{10}{2} + \cdots + \binom{10}{10} = 2^{10} - \left[\binom{10}{0} + \binom{10}{1}\right] = 1013$

(Q2) Find r if ${}^{10}C_r = {}^{10}C_{r+4}$? $r = 3$

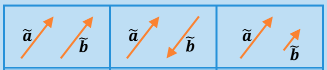
(Q3) Expand the brackets $(x^2 - 2)^5$:
$= \binom{5}{0}(x^2)^5(2)^0 - \binom{5}{1}(x^2)^4(2)^1 + \binom{5}{2}(x^2)^3(2)^2$
$- \binom{5}{3}(x^2)^2(2)^3 + \binom{5}{4}(x^2)^1(2)^4 - \binom{5}{5}(x^2)^0(2)^5$
$= (1)(x^{10})(1) - (5)(x^8)(2) + (10)(x^6)(4)$
$- (10)(x^4)(8) + (5)(x^2)(16) - (1)(1)(32)$
$= x^{10} - 10x^8 + 40x^6 - 80x^4 + 80x^2 - 32$

2-D VECTORS

CONCEPT OF A VECTOR

Scalar vs. Vector Quantities

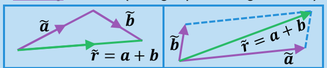
- Scalar: a quantity with only magnitude.
  - Distance and speed are scalar quantities.
- Vector: a quantity that has both magnitude and direction (represented by an arrow).
  - Displacement and velocity are vectors.



Same mag. Same dir. $(a = b)$	Same mag. Different dir. $(a = -b)$	Diff mag. Same dir. $(a \neq b)$

Vector Arithmetic ($\tilde{a} = xi + yj$)

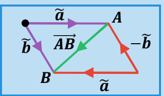
- Vectors notation separates into a horizontal component (i) and vertical component (j).
- Vectors can be written in form of $\langle x, y \rangle$ or $\binom{x}{y}$.
- Adding vectors (triangle/parallelogram rule):



Triangle Rule	Parallelogram Rule

- Subtracting vectors: to find vector from $\tilde{a}$ to $\tilde{b}$, reverse vector $\tilde{a}$ and subtract it from $\tilde{b}$.

$\overrightarrow{AB} = \tilde{b} - \tilde{a}$
- $\overrightarrow{AB}$: vector that goes from $\tilde{a}$ to $\tilde{b}$.



- Add, subtract and scalar multiply vectors:

$\binom{a}{b} \pm \binom{c}{d} = \binom{a \pm c}{b \pm d}$ $k \times \binom{a}{b} = \binom{k \times a}{k \times b}$

- k: scalar multiplier (i.e. a number).

Vector Arithmetic Examples

(Q1) Let $\tilde{a} = \langle 2, -1 \rangle$, $\tilde{b} = \langle 5, 2 \rangle$, $\tilde{c} = \langle -4, 3 \rangle$:

(Q1a) Determine the resultant vector $\tilde{a} + \tilde{c}$:
$\tilde{a} + \tilde{c} = \binom{2}{-1} + \binom{-4}{3} = \binom{2 + (-4)}{(-1) + 3} = \binom{-2}{2}$

(Q1b) Determine the vector $\tilde{b} - \tilde{c}$:
$\tilde{b} - \tilde{c} = \binom{5}{2} - \binom{-4}{3} = \binom{5 - (-4)}{2 - 3} = \binom{9}{-1}$

(Q1c) Determine the resultant vector $2\tilde{a} + 3\tilde{b}$:
$= 2\binom{2}{-1} + 3\binom{5}{2} = \binom{2 \times 2 + 3 \times 5}{2 \times -1 + 3 \times 2} = \binom{19}{4}$

VECTOR RULES

Vector Rules and Notation

- Magnitude ($|\tilde{a}|$) of a vector (i.e. length):

$\tilde{a} = xi + yj$ $|\tilde{a}| = \sqrt{x^2 + y^2}$

- Magnitude between two vectors ($\overrightarrow{AB}$):

$|\overrightarrow{AB}| = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2}$

- Unit vector ($\hat{a}$): vector with the same direction as $\tilde{a}$ but with a magnitude of 1.

$\hat{a} = a \div |a| = a/|a|$ $|\hat{a}| = 1$

- Dot product ($a.b$): returns a scalar result.

$a.b = (x_a \times x_b) + (y_a \times y_b)$

$a.b = b.a$ $a.a = |a|^2$ $a.b = |a||b|cos\theta$

- θ : the angle between vectors $\tilde{a}$ and $\tilde{b}$.

- Collinear points: 3 or more vectors on a line.

If $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ then A, B, C are collinear

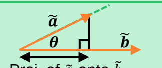
- Parallel and perpendicular vectors:

Parallel Vectors	Perpendicular Vectors
$\tilde{a} \parallel \tilde{b}$ if $\tilde{a} = k\tilde{b}$	$\tilde{a} \perp \tilde{b}$ if $\tilde{a}.\tilde{b} = 0$

- Vector projections: projecting of vector $\tilde{a}$ on $\tilde{b}$.

$|a|cos\theta\hat{b} = (a.\hat{b})\hat{b}$

$\dfrac{a.b}{b.b}b = \dfrac{a.b}{|b|^2}b$



Proj. of $\tilde{a}$ onto $\tilde{b}$

Common Vector Examples

(Q1) $\tilde{a} = \binom{3}{-1}$, $\tilde{b} = \binom{2}{6}$, $\tilde{c} = \binom{9}{-3}$, $\tilde{d} = \binom{4}{5}$:

(Q1a) Determine the magnitude of vector $\tilde{a}$:
$\tilde{a} = \langle 3, -1 \rangle \rightarrow |\tilde{a}| = \sqrt{3^2 + (-1)^2} = \sqrt{10} = 3.16$

(Q1b) Determine the unit vector for vector $\tilde{a}$:
$\tilde{a} = \langle 3, -1 \rangle \rightarrow \hat{a} = \dfrac{\tilde{a}}{|\tilde{a}|} = \dfrac{1}{\sqrt{10}}\binom{3}{-1} = \binom{3/\sqrt{10}}{-1/\sqrt{10}}$

(Q1c) Prove that $\tilde{a}$ and $\tilde{c}$ are parallel:
$\tilde{a} \parallel \tilde{c}$ if $\tilde{a} = k\tilde{c} \rightarrow \binom{3}{-1} = k\binom{9}{-3} \rightarrow k = \frac{1}{3}$

(Q1d) Prove that $\tilde{a}$ and $\tilde{b}$ are perpendicular:
$\tilde{a}.\tilde{b} = \binom{3}{-1}\cdot\binom{2}{6} = 3 \times 2 + (-1) \times 6 = 0$

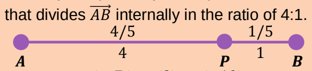
(Q1e) Find the angle between vectors $\tilde{c}$ and $\tilde{d}$:
$|\tilde{c}| = \sqrt{9^2 + 3^2} = \sqrt{90}$ $cos\theta = \tilde{c}.\tilde{d} \div (|\tilde{c}||\tilde{d}|)$
$|\tilde{d}| = \sqrt{4^2 + 5^2} = \sqrt{41}$ $cos\theta = 21 \div \sqrt{90}\sqrt{41}$
$\tilde{c}.\tilde{d} = 9 \times 4 - 3 \times 5 = 21$ $cos\theta = 0.3457, \theta = 70°$

(Q1f) Find the vector projection of $\tilde{a}$ onto $\tilde{d}$:
$\dfrac{a.d}{|d|^2}d = \dfrac{3 \times 4 - 1 \times 5}{(\sqrt{4^2 + 5^2})^2}\binom{4}{5} = \dfrac{12}{41}\binom{4}{5} = \binom{48/41}{60/41}$

POSITION VECTORS

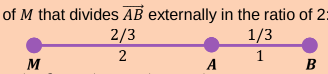
Position vectors Examples

(Q1) A & B have position vectors of $\langle 3i + 5j \rangle$ and $\langle -7i - 10j \rangle$ respectively. Find position vector of P that divides $\overrightarrow{AB}$ internally in the ratio of 4:1.



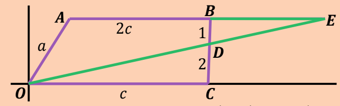
$\overrightarrow{AB} = \tilde{b} - \tilde{a} = \binom{-7}{-10} - \binom{3}{5} = \binom{-10}{-15}$
$\overrightarrow{AP} = \frac{4}{5} \times \overrightarrow{AB} = \frac{4}{5} \times \binom{-10}{-15} = \binom{-8}{-12}$ *Solve for $\tilde{p}$
$\overrightarrow{AP} = \tilde{p} - \tilde{a} \rightarrow \tilde{p} = \binom{-8}{-12} - \binom{3}{5} = \binom{-5}{-7}$

(Q2) A & B have position vectors of $\langle 4i - 10j \rangle$ and $\langle 10i + 20j \rangle$ respectively. Find position vector of M that divides $\overrightarrow{AB}$ externally in the ratio of 2:3.



$\overrightarrow{MA} = \frac{2}{3} \times \overrightarrow{MB} \rightarrow 3\overrightarrow{MA} = 2\overrightarrow{MB}$
$\therefore 3(\tilde{a} - \tilde{m}) = 2(\tilde{b} - \tilde{m}) \rightarrow \tilde{m} = 3\tilde{a} - 2\tilde{b}$
$= 3\binom{4}{-10} - 2\binom{10}{20} = \binom{12}{-30} - \binom{20}{40} = \binom{-8}{-70}$

(Q3) Trapezium $OABC$ has $\overrightarrow{OA} = a$, $\overrightarrow{OC} = c$ and $\overrightarrow{AB} = 2c$. D is point on $\overrightarrow{CB}$ such that $\overrightarrow{CD} = \frac{2}{3}\overrightarrow{CB}$. $\overrightarrow{OD}$ continued meets $\overrightarrow{AB}$ continued at E. If $\overrightarrow{OE} = h\overrightarrow{OD}$ and $\overrightarrow{AE} = k\overrightarrow{AB}$, find the value of h and k.



- Finding expressions for $\overrightarrow{CB}$, $\overrightarrow{CD}$ and $\overrightarrow{OD}$:

$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AB} = -c + a + 2c = a + c$
$\overrightarrow{CD} = \frac{2}{3}\overrightarrow{CB} = \frac{2}{3}(a + c) = \frac{2}{3}a + \frac{2}{3}c$
$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = c + \frac{2}{3}a + \frac{2}{3}c = \frac{2}{3}a + \frac{5}{3}c$

- Finding $\overrightarrow{OE}$ and solving for h and k:

$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE}$, using this to sub into $\overrightarrow{OE} = h\overrightarrow{OD}$ gives $h\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AE} = \overrightarrow{OA} + k\overrightarrow{AB}$. Solving:
$h\left(\frac{2}{3}a + \frac{5}{3}c\right) = a + k(2c) \rightarrow h = \frac{3}{2}$ and $k = \frac{5}{4}$

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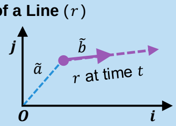
RELATIVE VECTORS

Vector Equation of a Line (r)

- A vector line (r) has initial position vector and then moves according to its velocity vector.



$r = position + \lambda \times direction = \tilde{a} + \lambda\tilde{b}$
• λ: scalar multiplier variable (e.g. time). • $\tilde{a}$: position vector of the object. • $\tilde{n}$: direction vector of the object.

Relative Vectors (${}_{A}r_{B}$ and ${}_{A}V_{B}$)

- Position of vector $\tilde{a}$ relative to vector $\tilde{b}$ (a.k.a. position of $\tilde{a}$ as seen from point of view of $\tilde{b}$).

${}_{A}R_{B} = R_A - R_B$
• ${}_{A}R_{B}$: position vector of $\tilde{a}$ relative to $\tilde{b}$. • R_A or R_B : position vector of $\tilde{a}$ or $\tilde{b}$.

- Velocity of vector $\tilde{a}$ relative to vector $\tilde{b}$ (a.k.a. velocity of $\tilde{a}$ as seen from point of view of $\tilde{b}$).

${}_{A}V_{B} = V_A - V_B$
• ${}_{A}V_{B}$: velocity vector of $\tilde{a}$ relative to $\tilde{b}$. • V_A or V_B : velocity vector of $\tilde{a}$ or $\tilde{b}$.

Collision of Two Vectors

- Collision or miss of two vectors is given by:

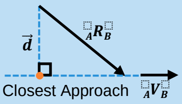
Collision	Miss
${}_{A}R_{B} = t \times {}_{B}V_{A}$	${}_{A}R_{B} \neq t \times {}_{B}V_{A}$

- Finding collision or miss of 2 moving vectors:

Step 1	Find vector equations of 2 lines in form of: $X = a + tb$ and $Y = c + td$
Step 2	Using these vectors, determine: ${}_{A}R_{B} = R_A - R_B$ and ${}_{B}V_{A} = V_B - V_A$
Step 3	Solve ${}_{A}R_{B} = t \times {}_{B}V_{A}$ for t. If t is found, collision occurs or if it is impossible to solve, vectors miss.

Closest Approach Between 2 Vectors

- If two vectors miss, there is a moment when they are closest to each other.



$\tilde{d} \,.\, {}_{A}V_{B} = 0$ where $\tilde{d} = \overrightarrow{BA} + ({}_{A}V_{B})t$
• $\tilde{d}$: distance of the closest approach.

- Finding time & distance of closest approach:

Step 1	Find the position vector $\overrightarrow{BA} = \tilde{a} - \tilde{b}$ and relative velocity ${}_{A}V_{B} = V_A - V_B$
Step 2	Find and simplify closest distance vector $\tilde{d} = \overrightarrow{BA} + ({}_{A}V_{B})t$
Step 3	Solve the dot product $\tilde{d} \,.\, {}_{A}V_{B} = 0$ for t by expanding and rearranging.
Step 4	Sub t into $\lvert\tilde{d}\rvert$ to find the closest distance and into $\tilde{a}$ or $\tilde{b}$ to find the location of closest approach.

Relative Velocity Examples

(Q1) Given that $V_A = \langle -5,6 \rangle$, $\lvert V_B \rvert = \sqrt{10}$ and $\lvert {}_{A}V_{B} \rvert = \sqrt{89}$, find the velocity vector V_B.

Let $V_B = \langle x, y \rangle$ which gives $x^2 + y^2 = 10$

${}_{A}V_{B} = V_A - V_B = (-5-x)^2 + (6-y)^2 = 89$

Simultaneously solve gives $x = 3$ and $y = 1$.

(Q2) Particle A is at $\langle -60,100 \rangle$ with a velocity of $\langle 20,-10 \rangle$ and particle B is at $\langle -20,-60 \rangle$ with a velocity of $\langle 10,30 \rangle$. Do particles collide?

$A = \binom{-60}{100} + \binom{20}{-10}t, B = \binom{-20}{-60} + \binom{10}{30}t$

- Finding relative vectors ${}_{A}R_{B}$ and ${}_{B}V_{A}$:

${}_{A}R_{B} = R_A - R_B = \binom{-60}{100} - \binom{-20}{-60} = \binom{-40}{160}$

${}_{B}V_{A} = V_B - V_A = \binom{10}{30} - \binom{20}{-10} = \binom{-10}{40}$

- Solving equation ${}_{A}R_{B} = t \times {}_{B}V_{A}$ for t:

${}_{A}R_{B} = t \times {}_{B}V_{A} \rightarrow \binom{-40}{160} = t \times \binom{-10}{40} \rightarrow t = 4$

Solution for t is possible, ∴ A and B collide.

(Q3) Particle A is at $\langle 100,150 \rangle$ with a velocity of $\langle 10,-5 \rangle$ and particle B is at $\langle 50,25 \rangle$ with a velocity of $\langle -5,20 \rangle$. What, when and where is closest approach of these particles?

$A = \binom{100}{150} + \binom{10}{-5}t, B = \binom{50}{25} + \binom{-5}{20}t$

- Finding relative vectors $\overrightarrow{BA}$ and ${}_{A}V_{B}$:

$\overrightarrow{BA} = \tilde{a} - \tilde{b} = \binom{100}{150} - \binom{50}{25} = \binom{50}{125}$

${}_{A}V_{B} = V_A - V_B = \binom{10}{-5} - \binom{-5}{20} = \binom{15}{-25}$

- Finding dist. equation $\tilde{d} = \overrightarrow{BA} + ({}_{A}V_{B})t$:

$\tilde{d} = \overrightarrow{BA} + ({}_{A}V_{B})t = \binom{50}{125} + \binom{15}{-25}t$

- Solving dot product $\tilde{d} \,.\, {}_{A}V_{B} = 0$ for t:

$\tilde{d} \,.\, {}_{A}V_{B} = \left[\binom{50}{125} + \binom{15}{-25}t\right] \cdot \binom{15}{-25} = 0$

Expanding gives: $\binom{50+15t}{125-25t} \cdot \binom{15}{-25} = 0$

$750 + 225t - 3125 + 625t = 0 \rightarrow t = 2.794$

- Finding distance $\lvert\tilde{d}\rvert$ and location:

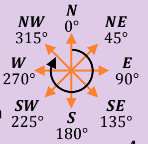
$\lvert\tilde{d}\rvert = \left\lvert\binom{50}{125} + 2.794\binom{15}{-25}\right\rvert = 107.18$ units

Location ($\tilde{a}$): $\binom{100}{150} + 2.794\binom{10}{-5} = \binom{127.94}{136.03}$

COMPONENT FORM VECTORS

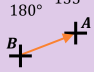
Bearing Notation and Rules

- True bearing (°T): angle between 0 and 360 that is measured clockwise around the compass starting from true north.



- Finding the bearing from one point to another:

Bearing of A from B = Start at B and go to A



- Find angle of elevation/depression of vector:

$xi + by \rightarrow \theta = tan^{-1}(y \div x)$

- Find components of vector with magnitude M and angle of elevation/depression θ:

$x = M \times cos(\theta)$	$y = M \times sin(\theta)$

Component Form Vector Examples

(Q1) Express the following in $xi + yj$ form:

$50km/h$ at 120°T:	$20km/h$ at 350°T:
Ref. angle = 30°	Ref. angle = 80°
$x = 50cos(30) = 43.3$	$x = 20cos(80) = 3.5$
$y = 50sin(30) = 25$	$y = 20sin(80) = 19.7$
In the 2nd quadrant:	In the 4th quadrant:
$43.3i - 25j$	$-3.5i + 19.7j$

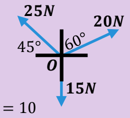
(Q2) Find the vector that will place the following system of vectors in equilibrium:

- Equating horizontal (i):

$-25cos45 = -17.678$

$20cos30 = 17.32$

$-17.678 + 17.32 = -0.357$



- Equating vertical (j):

$25sin45 = 17.678$, $20sin30 = 10$

$-15sin90 = -15$, $17.678 + 10 - 15 = 12.678$

- Finding vector to place in equilibrium:

$-0.357i + 12.678j$ is the resultant vector, to put the system in equilibrium the vector needs to be in opposite direction: $0.357i - 12.678j$

- Finding bearing of equilibrium vector:

$\theta = tan^{-1}(12.678/0.357) = 88.39°$ and vector is in 2nd quadrant hence $90 + 88.4 = 178.4°T$

(Q3) A yacht travels $150m/min$ in still waters. The captain wants to travel $300m$ east and $500m$ south of its position. Water current ($\tilde{c}$) is flowing at $20m/min$ from the north east. What direction should the captain aim and how long will the journey take to get to destination?

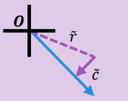
- Creating expressions:

Let $\tilde{r} = ai + bj$, finding $\tilde{c}$:

$x = y = 20cos(45) = 10\sqrt{2}$

$\tilde{c}$ direction is in 3rd quadrant:

∴ $\tilde{c} = -10\sqrt{2}i - 10\sqrt{2}j$



- Find equation to solve for a & b: $\binom{300}{-500}$

$\lvert\tilde{r}\rvert = \lvert ai + bj \rvert = 150 \rightarrow a^2 + b^2 = 150^2$

- Find another equation to solve for a & b:

$(ai + bj) - (10\sqrt{2}i + 10\sqrt{2}j) = \lambda(300i - 500j)$

$(a - 10\sqrt{2}) = 300\lambda$ and $(b - 10\sqrt{2}) = 500\lambda$

∴ $\lambda = \frac{a-10\sqrt{2}}{300} = \frac{b-10\sqrt{2}}{500}$ gives another equation.

- Simultaneously solve equations for a & b:

$a = 93.2$ and $b = -117.6 \rightarrow \tilde{r} = 93.2i - 117.6j$

- Finding bearing, resultant and time:

$\theta = tan^{-1}(117.6/93.2) = 51.6°$ and vector is in 2nd quadrant hence $90 + 88.4 = 141.6°T$

Resultant: $\binom{93.2}{-117.6} + \binom{-10\sqrt{2}}{-10\sqrt{2}} = \binom{79}{-131.7}$

Speed: $\left\lvert\binom{79}{-131.7}\right\rvert = 156.59m/min$

Distance to destination: $\left\lvert\binom{300}{-500}\right\rvert = 583.1km$

Journey time: $t = d \div t = 3.72\ min = 3m\ 43s$

ALGEBRAIC PROOFS

NATURE OF PROOFS

Proof Notation

Name		Example
Implication	$\Longrightarrow$	$A \Longrightarrow B$: If A then B
Equivalence	$\Leftrightarrow$	$A \Leftrightarrow B$: $A, B \Longrightarrow A$
For all	$\forall$	$\forall$ real numbers $x, x^2 \geq 0$
There exists	$\exists$	$\exists$ an even prime

Types of Alternative Conjectures

- For an implication ($A \Longrightarrow B$: If A then B), the following alternatives can be presented:

Name	Definition
Negative	If A then not B
Contrapositive*	If not B then not A
Converse**	If B then A
Inverse**	If not A then not B

- *Always true for a true statement.
- **Not always true for a true statement.

Nature of Proofs Examples

(Q1) Find a counterexample for the conjecture "doubling any number makes it larger"?

Not true for negative numbers or zero.

(Q2) Is the converse true for conjecture "if an animal is an ape then the animal is a mammal". Converse: if animal is mammal then it's an ape which is **not** true (e.g. deers are mammal).

PROOF BY DEDUCTION

Deductive Proof Techniques

- Deductive proofs refer to using well known mathematical principles to prove results.
- Methods of common numerical proofs:

Instance	Technique
Even Numbers	$2n$ for integers
Odd Numbers	$2n + 1$ for integers
2-Digit Numbers	$10A + B$
3-Digit Numbers	$100A + 10B + C$
Prove $a > b$	Show $a - b > 0$

Algebraic Deductive Proof Examples

(Q1) Prove that the converse and inverse of the conjecture is true: If $sin\theta = 1$ then $cos\theta = 0$

- **Converse**: If $cos\theta = 0$ then $sin\theta = 1$

$cos\theta = 0 \rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ ... which $sin\theta = 1$.

- **Inverse**: If $cos\theta \neq 0$ then $sin\theta \neq 1$

$cos\theta \neq 0 \rightarrow \theta \neq \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ ... which $sin\theta \neq 1$.

(Q2) Prove that the sum of squares of two consecutive even numbers is a multiple of 4.

$= (2n)^2 + (2n+2)^2 = 4n^2 + 4n^2 + 8n + 4$

$= 8n^2 + 8n + 4 = 4(2n^2 + 2n + 1)$ because a factor of 4 can be pulled out ∴ **multiple of 4**.

(Q3) Prove that $(a + b) \div 2 > \sqrt{ab}$

$a + b > 2\sqrt{ab}$ $\quad a^2 - 2ab + b^2 > 0$

$(a + b)^2 > 4ab$ $\quad (a - b)^2 > 0$

$a^2 + 2ab + b^2 > 4ab$

This statement is always true as any number that is squared is always **positive**.

(Q4) Prove that a 4-digit palindrome (i.e. a number that's same forwards as it is backwards such as 1221 or 9339) is always divisible by 11.

$= 1000x + 100y + 10y + x$

$= 1001x + 110y$ $\quad$ Has factor of 11

$= 11(91x + 10y)$ $\quad$ ∴ **divisible by 11**.

(Q5) Prove that an even number added to an odd number is always an odd number.

$= Even + Odd$

$= 2n + (2m + 1)$

$= 2(n + m) + 1$

$= 2k + 1$

As expression can be factorized in form $2x + 1$, $\forall$ integers, $2x + 1$ will always produce an **odd number**.

PROOF BY CONTRADICTION

Proof by Contradiction (*Negation*)

- For an implication ($A \Longrightarrow B$: If A then B), then the negative conjecture is if A then not B.
  - If the negative of a conjecture is false, then the original conjecture must true instead.

Step 1	Assume that the statement that is being proved is false.
Step 2	Try to prove the opposite statement as true (i.e. the negation).
Step 3	Use algebraic techniques to find a contradiction in the proof.
Step 4	State that contradiction disproves the negation, hence the original statement must instead be true.

Proof by Contradiction Example

(Q1) Prove the sum of two positive numbers always results in a positive number.

- Prove the negation, sum is a negative:

Introduce two variables a and b as the two positive integers (i.e. $a > 0$ and $b > 0$).

Then $a + b < 0$ which rearranges to $a < -b$

- State the contradiction in the proof:

If a and b are positive, $a < -b$ indicates that a is less than a negative, make it negative itself. Hence, initial supposition is contradicted that two positive numbers sum to a negative, ∴ original proposition is true; sum is **positive**.

(Q2) Prove if $n^3 + 5$ is odd then n is even.

- Prove the negation, n is odd:

Introduce integers a and b such that $n^3 + 5 = 2a + 1$ and $n = 2b + 1$. Substitute this for n:

$2a + 1 = n^3 + 5 \rightarrow 2a + 1 = (2b + 1)^3 + 5$

$2a + 1 = 8b^3 + 3(2b)^2(1) + 3(2b)(1)^2 + 1^3 + 5$

$2a = 8b^3 + 12b^2 + 6b + 5$

Divide both sides by 2 and rearrange:

$a = 4b^3 + 6b^2 + 3b - 2.5$

$2.5 = a - 4b^3 - 6b^2 - 3b$

- State the contradiction in the proof:

2.5 is a non-integer rational number, while the RHS of the equation above produces integers. Hence, $n^3 + 5$ can't be an odd number ∴**even**.

(Q3) Prove that $\sqrt{2}$ is irrational.

- Prove the negation, $\sqrt{2}$ is rational:

In order for a number to be rational, it can be expressed as a simplified fraction a/b, $b \neq 0$

$\sqrt{2} = a/b \rightarrow 2 = a^2/b^2 \rightarrow 2b^2 = a^2$

Since $2b^2$ is even, a^2 is even and hence a must be even as $even \times even = even$.

Let $a = 2k \rightarrow 2b^2 = 4k^2 \rightarrow b^2 = 2k^2$

If $2k^2$ is even, b^2 is even and hence b is even.

- State the contradiction in the proof:

If a and b are even, a/b is not a simplified fraction which contradicts initial assumption.

Hence, $\sqrt{2}$ can't be expressed as a rational fraction, ∴ $\sqrt{2}$ is an **irrational number**.

PROOF BY INDUCTION

Proof by Induction

- Statement is true for all natural numbers by showing that subsequent iterations succeed.
- Use notation LHS (left hand side) and RHS (right hand side) to equate expressions.

Step 1	Verify that the conjecture is true for the initial value of n by substitution.
Step 2	Assume that the conjecture is true for the substitution $n = k$.
Step 3	Use algebraic techniques to show that the conjecture is also true for the substitution $n = k + 1$.
Step 4	Merge the results from steps 2 and 3 to prove conjecture inductively for all possible values of n.

Proof by Induction Examples

(Q1) Use mathematical induction to prove:

$1 + 2 + 3 + 4 + 5 + \cdots + n = \frac{n(n+1)}{2}$

- Verify that the statement is true for $n = 1$:

$LHS = 1$, $RHS = (1 \times 2)/2 = 1$, ∴ $LHS = RHS$

- Assume statement is true for $n = k$:

$1 + 2 + 3 + 4 + 5 + \cdots + k = \frac{k(k+1)}{2}$

- Show that statement is true for $n = k + 1$:

$1 + 2 + 3 + 4 + 5 + \cdots + (k + 1) = \frac{(k+1)((k+1)+1)}{2}$

$\frac{k(k+1)}{2} + (k + 1) = \frac{(k+1)(k+2)}{2}$ $\quad$ *Sub from $n = k$

$LHS = \frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

$= \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}, LHS = RHS$

∴ statement is **true** $\forall$ natural numbers $n, n \geq 1$

(Q2) Use mathematical induction to prove:

$2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n+1)}{2}$ for $n \geq 1$

- Verify that the statement is true for $n = 1$:

$LHS = 2$, $RHS = (1 \times 4)/2 = 2$, ∴ $LHS = RHS$

- Assume statement is true for $n = k$:

$2 + 5 + 8 + 11 + 14 + \cdots + (3k - 1) = \frac{k(3k+1)}{2}$

- Show that statement is true for $n = k + 1$:

$2 + 5 + 8 + \cdots + (3(k + 1) - 1) = \frac{(k+1)(3(k+1)+1)}{2}$

$\frac{k(3k+1)}{2} + (3k + 2) = \frac{(k+1)(3k+4)}{2}$

$LHS = \frac{k(3k+1)}{2} + (3k + 2) = \frac{k(3k+1)}{2} + \frac{2(3k+2)}{2}$

$= \frac{k(3k+1)+2(3k+2)}{2} = \frac{3k^2+k+6k+4}{2} = \frac{3k^2+7k+4}{2}$

Factorising numerator $= \frac{(k+1)(3k+4)}{2}, LHS = RHS$

∴ statement is **true** $\forall$ natural numbers $n, n \geq 1$

(Q3) Prove $3^{2n} - 1$ is divisible by 8 for $n \geq 1$:

- Verify that the statement is true for $n = 1$:

$3^{2\times1} - 1 = 3^2 - 1 = 9 - 1 = 8$ is divisible by 8.

- Assume that statement is true for $n = k$:

$3^{2k} - 1$ is also divisible by 8.

- Show that statement is true for $n = k + 1$:

$3^{2(k+1)} - 1$ $\quad$ *Reverse the multiplication

$3^{2k+2} - 1$ $\quad$ index law $a^m \times a^n = a^{m+n}$

$3^2 \times 3^{2k} - 1$ $\quad$ *Split $9m = 8m + 1m$

$9 \times 3^{2k} - 1$ $\quad$ *Use assumption for $n = k$ as

$8 \times 3^{2k} + 3^{2k} - 1$ $\quad$ a substitution for a multiple of

$8 \times 3^{2k} + 8a$ $\quad$ some integer, a.

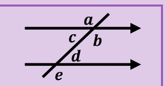
$8(3^{2k} + a)$ is a multiple of 8 hence divisible by 8.

∴ statement is **true** $\forall$ natural numbers $n, n \geq 1$

GEOMETRIC PROOFS

GEOMETRIC AXIOMS

Angle Axioms

Complementary (L) 2 angles add to 90°	
Supplementary (Y) $c + b = 180$	
Co-interior (C) $d + b = 180$	Corresponding (F) $e = b$
Alternate (Z) $c = d$	Vertical Opposite (Z) $a = b$

Interior Angle Axioms

Triangle	Circle
$a + b + c = 180°$	$a + b + c = 180°$
Quadrilateral	n-Sided Polygon
$a + b + c + d = 360°$	$sum = 180(n-2)°$

Congruent and Similar Triangles

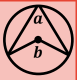
Congruent Triangle Proofs	SSS (Side/Side/Side)
SAS (Side/Angle/Side)	RHS (Right Angle/ Hypotenuse/Side)
AAS (Angle/Angle/Side)	ASA (Angle/Side/Angle)
Similar Triangle Proofs	AAA (Angle/Angle/Angle)
Proportional SSS (Side/Side/Side)	

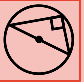
ATAR Math Specialist
Units 1 & 2 Exam Notes

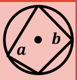
CIRCLE AXIOMS

Circle Axioms Axioms

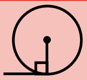
- The following axioms can be used in proofs:

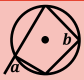
Angles at Centre and Circumference
(1) $2a = b$


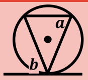
Angle inside a Semicircle
(2) $a = 90°$


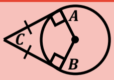
Opposite Angles in Cyclic Quadrilateral
(3) $a + b = 180$


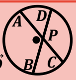
Angles in the Same Segment
(4) $a = b$

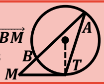

Angles between Tangent & Radius
(5) $a = 90°$


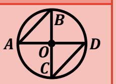
Exterior & Interior Opposite Angles
(6) $a = b$


Exterior angle in Opposite Segment
(7) $a = 90°$


Two Exterior Tangents to a Point
(8) $\overline{AC} = \overline{BC}$


Intersection of Two Chords
(9) $\overline{AP} \times \overline{PC} = \overline{BP} \times \overline{PD}$


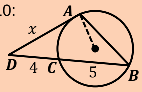
Secant Intersect with a Tangent
(10) $\overline{AM} \times \overline{BM} = \overline{TM}^2$


Two Chords of Equal Length and Equal Angles at the Centre
(11) If $\overline{AD} = \overline{BC}$ then $\overline{AB} = \overline{CD}$ and $\angle AOB = \angle COD$


CIRCLE PROOFS

Circle Proof Examples

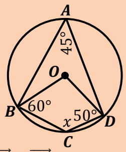
(Q1) Find the size of length x in the following:



- Using Circle Axiom #10:

$\overline{DC} \times \overline{DB} = \overline{AD}^2$
$\therefore 4 \times 9 = x^2$
$x^2 = 34, x = \sqrt{34} = 5.83$

(Q2) Find the size of angle x in the following:



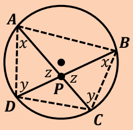
- Using Circle Axiom #1:

$2\angle BAD = \angle BOD$ (obtuse)
$\angle BOD = 2 \times 45 = 90°$

- Angles in Quadrilateral:

$60 + 50 + 90 + x = 360$
$x = 360 - 90 - 60 - 50$
$\therefore \angle DCB = x = 160°$

(Q3) Prove that $\overline{PA} \times \overline{PC} = \overline{PB} \times \overline{PD}$



- Using Circle Axiom #4:

$\angle DBP = \angle PBC = x$
$\angle BDP = \angle PCB = y$

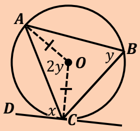
- Vertically Opposite:

$\angle BPD = \angle BPC = z$

- Similar Triangles:

$\Delta BPD \sim \Delta BPC\ (SSS)$
$\therefore \overline{PD}/\overline{PC} = \overline{PA}/\overline{PB} \to \overline{PA} \times \overline{PC} = \overline{PB} \times \overline{PD}$

(Q4) Prove circle axiom #7; angles in alternate segments are equal $x = y$:



- Using Circle Axiom #1:

$2\angle ABC = \angle AOC$ (obtuse)
$\angle AOC = 2 \times y = 2y$

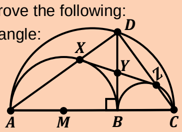
- Isosceles Triangles:

$\overline{AO} = \overline{CO}$ (circle radii)
$\angle OAC = \angle OCA = (180 - 2y)/2 = 90 - y$

- Using Circle Axiom #5:

$\angle OCD = \angle OCA + \angle ACD = 90°$
$\therefore 90 = x + 90 - y \to 0 = x - y \to x = y$

(Q5) Given that M is centre of a semi-circle with a diameter $\overline{AB}$ prove the following:



(Q5a) $BZDX$ is a rectangle:

- Circle Axiom #2:

The angles $\angle ADC$, $\angle AXB, \angle BZC = 90°$
$\therefore$ $BZDX$ is a rectangle as all of the interior angles are 90°.

(Q5b) ΔMYX is congruent to ΔMYB:

- From part (a), the lengths of the diagonals inside a rectangle are equal, $\overline{XY} = \overline{YB}$.
- Also, $\overline{MX} = \overline{MB}$ as both are radii in the small semicircle with diameter $\overline{AB}$.
- $\overline{MY}$ is common side to ΔMYX and ΔMYB.

$\therefore$ $\Delta MYX \cong \Delta MYB$ (SSS) Q.E.D.

(Q5c) The line drawn by $\overline{XZ}$ is a tangent to the semicircle with diameter $\overline{AB}$.

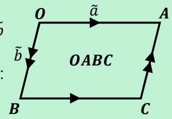
- $\angle YBM = 90°$ which is given in diagram.
- From part (b), if $\Delta MYX \cong \Delta MYB$ then $\angle YXM = \angle YBM = 90°$ (matching angles).
- This inherently proves circle axiom #5.

$\therefore$ $\overline{XZ} \perp \overline{AB} \to \overline{XZ}$ is a tangent to semi-circle.

VECTOR PROOFS

Vector Proof Examples

(Q1) Prove if the diagonals of a parallelogram are perpendicular, then it is a rhombus.



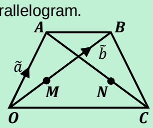
- Defining variables:

Let $\overrightarrow{OA} = \tilde{a}$ and $\overrightarrow{OB} = \tilde{b}$
Hence $\overrightarrow{AB} = \tilde{b} - \tilde{a}$

- Constructing proof:

If $\overrightarrow{AB} \perp \overrightarrow{OC}$ then must prove that $\overrightarrow{AB} \cdot \overrightarrow{OC} = 0$
$(\tilde{b} - \tilde{a}) \cdot (\tilde{a} + \tilde{b}) = \tilde{a}.\tilde{b} + \tilde{b}.\tilde{b} - \tilde{a}.\tilde{a} - \tilde{a}.\tilde{b} = 0$
$\tilde{b}.\tilde{b} - \tilde{a}.\tilde{a} = 0 \to \tilde{b}.\tilde{b} = \tilde{a}.\tilde{a} \to |\tilde{b}|^2 = |\tilde{a}|^2$
$\therefore |\tilde{b}| = |\tilde{a}|, \therefore OABC$ is a rhombus.

(Q2) $OABC$ is a trapezium with $\overrightarrow{OC} = 2\overrightarrow{AB}$. M lies on diagonal $\overrightarrow{OB}$ so that $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OB}$ and N lies on diagonal $\overrightarrow{CA}$ so that $\overrightarrow{CN} = \frac{1}{3}\overrightarrow{CA}$. Prove that $ABNM$ is in fact a parallelogram.



- Defining variables:

Let $\overrightarrow{OA} = \tilde{a}$ and $\overrightarrow{OB} = \tilde{b}$

- Defining all sides:

Find $\overrightarrow{OM}$: $\overrightarrow{OM} = \frac{1}{3}\tilde{b}$
Find $\overrightarrow{ON}$: $\overrightarrow{ON} = \overrightarrow{OC} + \overrightarrow{CN}$
$\overrightarrow{ON} = 2\overrightarrow{AB} + \frac{1}{3}\overrightarrow{CA} = 2\overrightarrow{AB} + \frac{1}{3}(\overrightarrow{CO} + \overrightarrow{OA})$
$= 2\tilde{b} - 2\tilde{a} + \frac{1}{3}(2\tilde{a} - 2\tilde{b} + \tilde{a}) = \frac{4}{3}\tilde{b} - a$
Find $\overrightarrow{MN}$: $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{4}{3}\tilde{b} - a - \frac{1}{3}\tilde{b} = \overrightarrow{AB}$
Find $\overrightarrow{AM}$: $\overrightarrow{AM} = \frac{1}{3}\tilde{b} - \tilde{a}$ *Use $\overrightarrow{AB} = \tilde{b} - \tilde{a}$
Find $\overrightarrow{BN}$: $\overrightarrow{BN} = \overrightarrow{BO} + \overrightarrow{ON} = -\tilde{b} + \frac{4}{3}\tilde{b} - a = \frac{1}{3}\tilde{b} - a$

- Constructing parallelogram proof:

$\overrightarrow{BN} = \overrightarrow{AM}$ and $\overrightarrow{MN} = \overrightarrow{AB}$, $\therefore$ pairs of congruent opposite sides $\therefore ABNM$ is a parallelogram.

TRIGONOMETRY

TRIGONOMETRIC FORMULAE

Exact Values of Trigonometric Ratios

Deg.	0°	30°	45°	60°	90°
Rad.	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
Sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
Cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
Tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	N/A

Pos/Neg of Trigonometric Ratios

- Positive trig ratios: All Stations To Central

Quad.	Q.1	Q.2	Q.3	Q.4	Unit Circle
Sin	+	+	−	−	2 1
Cos	+	−	−	+	3 4
Tan	+	−	+	−	

Range of Trigonometric Ratios

Sin	Cos	Tan
-1 to 1	-1 to 1	$-\infty$ to ∞

Trigonometric Identities

- Sum and difference identities:

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$
$$\tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$$

- Reciprocal identities:

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)} \quad \sec(x) = \frac{1}{\cos(x)} \quad \cot(x) = \frac{1}{\tan(x)}$$

- Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1 \quad 1 + \tan^2\theta = \sec^2\theta$$

- Quotient identities:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \cot(x) = \frac{\cos(x)}{\sin(x)}$$

- Co-function identities:

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x) \quad \cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$

- Parity identities (*i.e.* even and odd):

$$\sin(-x) = -\sin(x) \quad \cos(-x) = \cos(x)$$
$$\tan(-x) = -\tan(x) \quad \sec(-x) = \sec(x)$$

- Double angle identities:

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$
$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

- Combination angle identities:

$$\cos X \cos Y = \frac{1}{2}(\cos(X - Y) + \cos(X + Y))$$
$$\sin X \sin Y = \frac{1}{2}(\cos(X - Y) - \cos(X + Y))$$
$$\sin X \cos Y = \frac{1}{2}(\sin(X + Y) + \sin(X - Y))$$

- Power reducing identities:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \quad \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

TRIGONOMETRIC PROOFS

Trigonometric Identity Examples

(Q1) Prove these trigonometric identities:

(Q1a) $\sec(\theta)cosec(\theta)\cot(\theta) = 1 + \cot^2(\theta)$
$LHS = \frac{1}{\cos(\theta)} \times \frac{1}{\sin(\theta)} \times \frac{\cos(\theta)}{\sin(\theta)} = \frac{1}{\sin^2(\theta)}$
$RHS = 1 + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$
$\therefore RHS = LHS$ Q.E.D. by meeting halfway.

(Q1b) Prove $LHS = RHS$ for $\frac{1 - \sin(2\theta)}{\cos(2\theta)} = \frac{1 - \tan(\theta)}{1 + \tan(\theta)}$
$LHS = \frac{\cos^2(\theta) + \sin^2(\theta) - 2\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)}$ *Factorise and cancel
$LHS = \frac{(\cos\theta - \sin\theta)(\cos\theta - \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$
$RHS = \frac{1 - \frac{\sin(\theta)}{\cos(\theta)}}{1 + \frac{\sin(\theta)}{\cos(\theta)}} \times \frac{\cos(\theta)}{\cos(\theta)} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = LHS$ Q.E.D.

(Q1c) $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$
$LHS = \sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$
$LHS = \sin\theta\cos 2\theta + 2\sin\theta\cos^2\theta$
$LHS = \sin\theta(1 - 2\sin^2\theta) + 2\sin\theta(1 - \sin^2\theta)$
$LHS = \sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta$
$LHS = 3\sin(\theta) - 4\sin^3(\theta) = RHS$ Q.E.D.

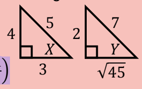
(Q1d) $\cos(4\theta) = 1 - 8\cos^2\theta + 8\cos^4\theta$
$LHS = \cos(2 \times 2\theta) = 2\cos^2(2\theta) - 1$ *Expand and simplify
$LHS = 2(\cos(2 \times 2\theta)) - 1$
$LHS = 2[(2\cos^2(\theta) - 1)(2\cos^2(\theta) - 1)] - 1$
$LHS = 2[4\cos^4(\theta) - 4\cos^2(\theta) + 1] - 1$
$LHS = 1 - 8\cos^2\theta + 8\cos^4\theta = RHS$ Q.E.D.

TRIGONOMETRIC ALGEBRA

Trigonometric Algebra Examples

(Q1) If $\cos(X) = 3/5$ and $\sin(Y) = 2/7$ over the interval $\pi/2 \le X, Y \le \pi$, find $\sin(X + Y)$:

- Finding $\sin(X + Y)$: *Triangle X and Y:



$= \sin X \cos Y + \sin Y \cos X$
$= \left(\frac{4}{5}\right)\left(\frac{\sqrt{45}}{7}\right) + \left(\frac{2}{7}\right)\left(\frac{3}{5}\right)$
$= \frac{4\sqrt{45} + 6}{35} = \frac{12\sqrt{5} + 6}{35} = 6\left(\frac{2\sqrt{5} + 1}{35}\right)$

(Q2) Solve $3\cos(x) + 4\sin(x) = 2, 0 \le x \le 360$

$$a\sin x \pm b\cos x = \sqrt{a^2 + b^2} \times \sin(x + \tan^{-1}(b/a))$$

$\therefore 3\cos(x) + 4\sin(x) = 5 \times \sin(x + 53.13)$
$\therefore \sin(x + 53.13) = 2/5 \to \sin(y) = 2/5$
$y = 23.58° \to \therefore x = -29.55 = 330.45°$

TRIGONOMETRIC FUNCTIONS

Period, Amplitude and Phase

- Period: how long it takes for a trigonometric function to complete 1 full cycle.
  - Period relates to 'b' in each equation:

Ratio	Sine	Cosine	Tangent
Period	2π	2π	π
b	2π/Period	2π/Period	π/Period
Ratio	Cosec	Sec	Cot
Period	2π	2π	π
b	2π/Period	2π/Period	π/Period

- Amplitude: maximum vertical distance in units from the x-axis to max/min points.
  - Amplitude relates to 'a' in each equation:

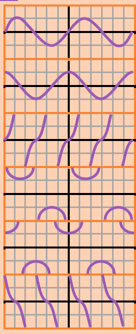
$$a = \frac{max\ y_{value} - min\ y_{value}}{2}$$

- Phase: refers to any left or rightward shifts.
  - Phase relates to 'c' in each equation.
  - Sine and Cosine have a phase shift of $\pi/2$:

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right) \quad \cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$$

- Vertical Shift: relates to 'd' in each equation.

Sine, Cosine and Tangent Functions



Sine $y = a\sin[b(x + c)] + d$

Cosine $y = a\cos[b(x + c)] + d$

Tangent $y = a\tan[b(x + c)] + d$

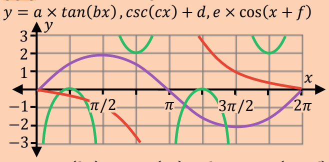
Cosecant $y = a\csc[b(x + c)] + d$

Secant $y = a\sec[b(x + c)] + d$

Cotangent $y = a\cot[b(x + c)] + d$

Trigonometric Function Examples

(Q1) Find the missing values in the functions:
$y = a \times \tan(bx), \csc(cx) + d, e \times \cos(x + f)$



$a\tan(bx)$ — $\csc(cx) + d$ — $e\cos(x + f)$

Tangent mirrored and period doubled, $a = -1$ and $b = 0.5$. Cosecant has period halved $c = -2$ and $d = 1$. Cosine $e = 2$ and $f = \pi/2$

MATRICES

MATRIX FORMULAE

Matrix Terminology

- A matrix (the plural is matrices) is an array (a.k.a. a grid) of numbers of a certain size.
- Matrix order/size: the number of rows and columns that are in a matrix.
  - When writing a matrix, the number of rows always goes first then number of columns.

$n \times m$ Matrix

- m : number of rows in a matrix.
- n : number of columns in a matrix.

- Matrix elements/entries (a_{ij}): represents the element i^{th} row and j^{th} column in matrix A.

$$3 \times 2 \text{ Matrix } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{22} \end{bmatrix}$$

Common Types of Matrices

$\begin{bmatrix} 1 & 4 & 1 \end{bmatrix}$	Row Matrix: consists of only one row.
$\begin{bmatrix} 5 \\ 9 \end{bmatrix}$	Column Matrix: consists of only one column.
$\begin{bmatrix} 2 & 6 \\ 5 & 3 \end{bmatrix}$	Square Matrix: a matrix of any size with a condition that # of rows = # of columns
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	Zero Matrix (0): a matrix of any size with 0 as all entries.
$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Identity Matrix (I_n): square matrix with all elements in the leading diagonal (goes from top left to bottom right) as 1 and all other entries as 0.

Matrix Arithmetic

- Adding and subtracting matrices: can only be possible if both matrices have same size.

$$A + B = a_{ij} + b_{ij} \quad A - B = a_{ij} - b_{ij}$$

- $a_{ij} \pm b_{ij}$: add/subtract matching entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

- Scalar multiplication: can use on any size.

$$kA = ka_{ij}$$

- k: scalar multiplier (*i.e.* a number).
- ka_{ij}: multiply all entries in matrix by k.

$$k \times \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} k \times a & k \times b & k \times c \\ k \times d & k \times e & k \times f \end{bmatrix}$$

- Multiplying matrices: multiply each element in row of 1st matrix with matching element from each column of 2nd matrix and add.

Matrix $A = m \times n$ & Matrix $B = p \times q$
$A \times B$ only possible if $n = p$
Matrix of size $m \times q$ is created

- E.g. 1: $(1 \times 3)(3 \times 1) = 1 \times 1$ Matrix

$$\begin{bmatrix} a & b & c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} ad + be + cf \end{bmatrix}$$

- E.g. 2: $(1 \times 3)(3 \times 2) = 1 \times 2$ Matrix

$$\begin{bmatrix} a & b & c \end{bmatrix} \times \begin{bmatrix} d & e \\ f & g \\ h & j \end{bmatrix} = \begin{bmatrix} ad + bf + ch & ae + bg + cj \end{bmatrix}$$

- E.g. 3: $(2 \times 2)(2 \times 2) = 2 \times 2$ Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Determinant and Inverse of a Matrix

- Determinant ($det(A)$ or $|A|$) of 2×2 matrix is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to det(A) = |A| = ad - bc$$

- Two types of determinants based on value:

$det(A) \ne 0$	Matrix A is non-singular and is able to be inverted.
$det(A) = 0$	Matrix A is singular and is not able to be inverted.

- Inverse (A^{-1}) of 2×2 matrix is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to A^{-1} = \frac{1}{det(A)}\begin{bmatrix} d & -b \\ -c & b \end{bmatrix}$$

- Properties of a matrix with its inverse:

$AA^{-1} = I$ and $A^{-1}A = I$ — If $AB = kI$ then $A^{-1} = \frac{1}{k}B$ and $B^{-1} = \frac{1}{k}A$

Transpose Matrices (A^T)

- Transpose matrices are reflected in leading diagonal so that rows and columns swap:

A is a $n \times m$ matrix $\to A^T$ is a $m \times n$ matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \to A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

ATAR Math Specialist
Units 1 & 2 Exam Notes

MATRIX ARITHMETIC

Common Rules of Matrix Arithmetic

$A+B=B+A$	$k(AB)=A(kB)$
$AA=A^2$	$AB\neq BA$
$IA=AI=A$	$0A=0$
$A(B\pm C)=AB\pm AC$	$(A\pm B)C=AC\pm BC$

Matrix Arithmetic Equations

(Q1) Given the following matrices, determine:
$A=\begin{bmatrix}5&1\\-1&7\end{bmatrix}$ $B=\begin{bmatrix}2\\5\end{bmatrix}$ $C=\begin{bmatrix}0&6\\2&2\end{bmatrix}$ $D=\begin{bmatrix}3&5&2\\1&0&4\end{bmatrix}$

(Q1a) $-2B$
$-2B=-2\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}-4\\-10\end{bmatrix}$

(Q1b) $5a_{21}-c_{22}\times d_{12}$
$(5\times-1)-2\times5=-5-2\times5=-15$

(Q1c) $A+C$
$\begin{bmatrix}5+0&1+6\\-1+2&7+2\end{bmatrix}=\begin{bmatrix}5&7\\1&9\end{bmatrix}$

(Q1d) BD **impossible**
$(2\times1)(2\times3)$ is not compatible (as $1\neq2$)

(Q1d) AD $(2\times2)(2\times3)$ is compatible and will produce a 2×3 matrix as the answer:
$\begin{bmatrix}5&1\\-1&7\end{bmatrix}\times\begin{bmatrix}3&5&2\\1&0&4\end{bmatrix}=\begin{bmatrix}16&25&14\\4&-5&26\end{bmatrix}$ working out:
$\begin{bmatrix}5\times3+1\times1&5\times5&5\times2+1\times4\\-1\times3+7\times1&-1\times5&-1\times2+7\times4\end{bmatrix}$

(Q1e) A^{-1}, the inverse of matrix A:
$A^{-1}=\frac{1}{5\times7-1\times-1}\begin{bmatrix}7&-1\\1&5\end{bmatrix}=\begin{bmatrix}7/36&-1/36\\1/36&5/36\end{bmatrix}$

(Q2) Rearrange equation making X the subject:

(Q2a) $AX+2X=B$ *Pre-multiply the inverse of brackets $(A+2I)$
$(A+2I)X=B$
$(A+2I)^{-1}(A+2I)X=(A+2I)^{-1}B$ *Simplify using rule $X^{-1}X=I$
$X=(A+2I)^{-1}B$

(Q2b) $XA+XB=C$ *Post-multiply inverse of the brackets $(A+B)$
$X(A+B)=C$
$X(A+B)(A+B)^{-1}=C(A+B)^{-1}$ *Simplify
$X=C(A+B)^{-1}$

(Q2c) $X^2-2X-3I=0$ *Ensure alone numbers have I matrix
$XX-2X=3I$ *Expand matrix $X^2=XX$
$(X-2I)X=3I$ *Post-multiply inverse of matrix X
$(X-2I)XX^{-1}=3X^{-1}$
$X-2I=3X^{-1}$ *Collect like terms on one side of the equation
$X^{-1}=\frac13X-\frac23I$

(Q3) Solve the following for matrix X:
$X\begin{bmatrix}1&-1\\-3&4\end{bmatrix}+X=\begin{bmatrix}4&-2\\5&-6\end{bmatrix}$
$X\left(\begin{bmatrix}1&-1\\-3&4\end{bmatrix}+\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}4&-2\\5&-6\end{bmatrix}$ *Factorise matrices
$X\begin{bmatrix}2&-1\\-3&5\end{bmatrix}=\begin{bmatrix}4&-2\\5&-6\end{bmatrix}$ *Post-multiply inverse of simplified matrix
$X=\begin{bmatrix}4&-2\\5&-6\end{bmatrix}\begin{bmatrix}2&-1\\-3&5\end{bmatrix}^{-1}$ *Simplify matrices
$X=\begin{bmatrix}4&-2\\5&-6\end{bmatrix}\times\frac17\begin{bmatrix}5&1\\3&2\end{bmatrix}=\frac17\begin{bmatrix}14&0\\7&-7\end{bmatrix}=\begin{bmatrix}2&0\\1&-1\end{bmatrix}$

(Q4) Find the values of a if X is singular:
$\begin{bmatrix}a+1&a+1\\-1&a-3\end{bmatrix}$ singular means $\det(X)=0$
$\therefore(a+1)(a-3)-(-1)(a+1)=0$
$a^2-2a-3+a+1=a^2-a-2=0$
$(a-2)(a+1)=0\to a=\mathbf{2}$ or $a=\mathbf{-1}$

(Q5) Given $A=\begin{bmatrix}1&2\\2&3\end{bmatrix}$ and assuming B is an invertible 2×2 matrix, find X if $AXB+2B=0$
- Rearranging equation for matrix X:

$\det(A)=1\times3-2\times2=1$
- Rearranging equation for matrix X:

$(AX+2I)B=0\to AXB=-2B\to AX=-2I$
$\therefore X=-2IA^{-1}=\begin{bmatrix}-2&0\\0&-2\end{bmatrix}\times\frac{1}{-1}\begin{bmatrix}3&-2\\-2&1\end{bmatrix}$
$X=\begin{bmatrix}-2&0\\0&-2\end{bmatrix}\begin{bmatrix}-3&2\\2&-1\end{bmatrix}=\begin{bmatrix}6&-4\\-4&2\end{bmatrix}$

(Q6) Given A,B and C are non-singular square matrices and $AB=BC$, show $A^2=BC^2B^{-1}$
$RHS=BCCB^{-1}=ABCB^{-1}=A^2I=\mathbf{A^2}=LHS$

SIMULTANEOUS EQUATIONS

Using Matrices to Solve Equations

Step 1	Rearrange both equations to be solved in the form of $ax+by=c$
Step 2	Create the matrix equation $AX=B$ where A is a 2×2 matrix of the coefficients, B is 2×1 of solutions and X is 2×1 matrix of variables.
Step 3	Rearrange matrix equation $AX=B$ to $X=A^{-1}B$ and solve for X.

Simultaneous Equations Examples

(Q1) The equations $7y=6x+4$ and $5x=3y$ can be expressed in the matrix form $AX=B$. Determine two of these matrices X and B:
- Rearrange equations in form $ax+by=c$:

$6x-7y=4$ and $5x-3y=0$
- Create equation in form of $AX=B$:

$\begin{bmatrix}-6&7\\3&-5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}\to X=\begin{bmatrix}x\\y\end{bmatrix},B=\begin{bmatrix}4\\0\end{bmatrix}$

(Q2) Using matrices solve the equations $y=-2x+8$ and $y=2+x$ simultaneously.
- Rearrange equations in form $ax+by=c$:

$2x+1y=8$ and $-1x+1y=2$
- Create equation $AX=B$ and solve:

$AX=B\to\begin{bmatrix}2&1\\-1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\2\end{bmatrix}$ *Rearrange and solve
$X=A^{-1}B\to\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2&1\\-1&1\end{bmatrix}^{-1}\begin{bmatrix}8\\2\end{bmatrix}=\frac13\begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}8\\2\end{bmatrix}$
$X=\frac13\begin{bmatrix}6\\12\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}\to x=2,y=4$

TRANSFORMATION MATRICES

Transformation Matrices Rules

- Matrix is pre-multiplied by a transformation matrix to alter its co-ordinates.

$TO=F$	$O=T^{-1}F$	$T=FO^{-1}$

- T : 2×2 transformation matrix.
- O : co-ordinate matrix of original image.
- F : co-ordinate matrix of final image.
- T^{-1} : returns final image to original.
- Creating a 2×2 transformation matrix that combines multiple individual transformations:

$$T=\cdots\times C\times B\times A$$

- A : first individual transformation matrix.
- $C\ldots$: final individual transformation.
- Area of final image after a transformation:

$$A_{Final}=|\det(T)|\times A_{Original}$$

- A : area of the final and original image.
- $|\det(T)|$: absolute value of determinant.

Types of Transformation Matrices

- Reflection transformation matrices:

$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$	Reflection about the x-axis.
$\begin{bmatrix}-1&0\\0&1\end{bmatrix}$	Reflection about the y-axis.
$\begin{bmatrix}0&1\\1&0\end{bmatrix}$	Reflection about the equation $y=x$ (i.e. 45° line).
$\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$	Reflection about the equation $y=-x$ (i.e. −45° line).

- Rotation transformation matrices:

$\begin{bmatrix}0&1\\-1&0\end{bmatrix}$	Rotation 90° clockwise about the origin.
$\begin{bmatrix}0&-1\\1&0\end{bmatrix}$	Rotation 90° anti-clockwise about the origin.
$\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$	Rotation 180° clockwise about the origin.
$\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}$	Rotation $\theta°$ anti-clockwise about the origin.

- Dilation transformation matrices:

$\begin{bmatrix}k&0\\0&1\end{bmatrix}$	Dilation factor where $k>0$ in the direction of the x-axis.
$\begin{bmatrix}1&0\\0&k\end{bmatrix}$	Dilation factor where $k>0$ in the direction of the y-axis.
$\begin{bmatrix}k&0\\0&k\end{bmatrix}$	Enlargement factor where $k>0$ about the origin.
$\begin{bmatrix}1&k\\0&1\end{bmatrix}$	Shear factor k in the direction of the positive x-axis.
$\begin{bmatrix}1&0\\k&1\end{bmatrix}$	Shear factor k in the direction of the positive y-axis.

Transformation Matrices Examples

(Q1) The image of (a,b) under transformation of $T=\begin{bmatrix}6&-3\\2&4\end{bmatrix}$ is (a,b). Find values of a and b.
$\begin{bmatrix}6&-3\\2&4\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}4\\0\end{bmatrix}\to\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}6&-3\\2&4\end{bmatrix}^{-1}\begin{bmatrix}4\\0\end{bmatrix}$
Solving this equation gives $a=2$ and $b=1$

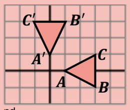
(Q2) Find the final image of the parallelogram $A(-1,1)$, $B(2,7)$, $C(4,3)$ and $D(1,-3)$ after it has been rotated 90° clockwise about origin:
$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}-1&2&4&1\\1&7&3&-3\end{bmatrix}=\begin{bmatrix}1&7&3&-3\\1&-2&-4&-1\end{bmatrix}$

(Q3a) Rectangle with vertices $A(1,0)$, $B(3,0)$, $C(3,3)$ and $D(1,3)$ undergoes transformations:
- 1: Rotate 90° anti-clockwise about origin
- 2: Shear dilation factor 2 to the y-axis
- 3: enlargement by a scale factor of $\sqrt3$

Find matrix that transforms $ABCD$ to $A'B'C'D'$.
Matrix order: enlargement → shear → rotate
Matrix: $\begin{bmatrix}\sqrt3&0\\0&\sqrt3\end{bmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ and multiply from right $=\begin{bmatrix}\sqrt3&0\\0&\sqrt3\end{bmatrix}\begin{bmatrix}0&-1\\1&-2\end{bmatrix}=\begin{bmatrix}0&-\sqrt3\\\sqrt3&-2\sqrt3\end{bmatrix}$

(Q3b) Find area of the final image $A'B'C'D'$:
Area of original: $(3-0)(3-1)=3\times2=6$
$|\det(T)|$: $|(0\times-2\sqrt3)-(-\sqrt3\times\sqrt3)|=3$
Area of final image $A'B'C'D'$: $6\times3=\mathbf{18\ units^2}$

(Q4a) Find transformation of ABC to $A'B'C'$:



From the diagram, it is clear that ABC has been rotated anti-clockwise by 90° about the origin which gives: $T=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$

(Q4b) Find matrix of the 2nd transformation: $A'(0,1)\to A''(0,2)$ and $B'(1,3)\to B''(3,6)$
$T\begin{bmatrix}0&1\\1&3\end{bmatrix}=\begin{bmatrix}0&3\\2&6\end{bmatrix}\to T=\begin{bmatrix}0&3\\2&6\end{bmatrix}\begin{bmatrix}-3&1\\1&0\end{bmatrix}=\begin{bmatrix}3&0\\0&2\end{bmatrix}$
Hence, 2nd transformation is dilation along x-axis of 3 and dilation along y-axis of 2.

(Q4c) Calculate area of the triangle $A''B''C''$:
Area of $A'B'C'$: $0.5\times2\times2=2\ units^2$
$|\det(T)|$: $|(3\times2)-(0\times0)|=6$
Area of image $A''B''C''$: $6\times2=\mathbf{12\ units^2}$

(Q4d) What is the transformation matrix that will transform final image $A''B''C''$ back to ABC:
$\begin{bmatrix}3&0\\0&2\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}A=A''$ $\therefore A=\begin{bmatrix}0&-3\\2&0\end{bmatrix}^{-1}\times A''$
$\begin{bmatrix}0&-3\\2&0\end{bmatrix}A=A''$ $A=\begin{bmatrix}0&1/2\\-1/3&0\end{bmatrix}\times A''$

TYPES OF NUMBERS

DECIMAL CONVERSIONS

Nature of Recurring Decimals

- Convert any recurring decimal to a fraction:

Step 1	Let the recurring decimal to be converted to a fraction be x.
Step 2	Multiply x by a power of 10 so that the decimal point galls directly to the left of the first repeating value.
Step 3	Multiply x by a power of 10 so that decimal point galls directly to the right of first set of repeating values.
Step 4	Subtract the equation found in Step 2 from equation found in Step 3.
Step 5	Solve the equation for n; this is the converted fraction from a decimal.

Decimal Conversion Examples

(Q1) Write 0.5555 … as a simplified fraction.
$x=0.\dot5$ $10n-n=5.\dot5-0.\dot5$
$10n=5.\dot5$ $9n=5$ $n=\frac59$

(Q2) Write 0.2343434 … as a simplified fraction.
$x=0.2\overline{34}$ $1000n-10n=234.\overline{34}-2.\overline{34}$
$10n=2.\overline{34}$ *Subtract equations $990n=232$
$1000n=234.\overline{34}$ $n=232/990$

(Q3) Write 2.0199999 … as a simplified fraction.
$x=2.01\dot9$ $1000n-100n=2019.\dot9-201.\dot9$
$100n=201.\dot9$ *Subtract equations $900n=1818$
$1000n=2019.\dot9$ $n=1818/900$

IMAGINARY NUMBERS

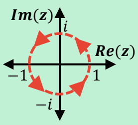
Concept of an Imaginary Number

- Imaginary numbers (i) aren't considered real numbers and enables solutions to equations such as $x^2=-16$ possible (i.e. $x=4i$).

$i=\sqrt{-1}$	$i^2=i\times i=\sqrt{-1}\times\sqrt{-1}=-1$

Powers of Imaginary Numbers (i)

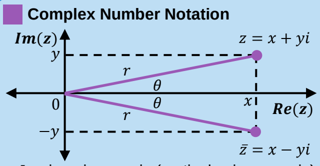
$i^{-4}=1$	$i^0=1$	$i^4=1$
$i^{-3}=\sqrt{-1}$	$i^1=\sqrt{-1}$	$i^5=\sqrt{-1}$
$i^{-2}=-1$	$i^2=-1$	$i^6=-1$
$i^{-1}=-i$	$i^3=-i$	$i^7=-i$



- To find value of i^n, divide power by 4:
  - Remainder 0 = 1
  - Remainder 1 = i
  - Remainder 2 = −1
  - Remainder 3 = $-i$

COMPLEX NUMBER RULES

Complex Number Notation



- Im: imaginary axis (vertical axis → y-axis).
- Re: real axis (horizontal axis → x-axis).
- z: complex number ($z=x+yi$).
- $\bar z$: conjugate of a complex number ($\bar z=x-yi$) and is reflected in the real axis.
- x: real components (horizontal axis).
- y: imaginary component (vertical axis).
- r: modulus (length) of a complex number and can also be represented by $|z|$.
- θ: argument (angle that the complex number makes with the real axis) of complex number and can also be represented by $\arg(z)$.

Complex Number Rules

- Rules for Complex Conjugates:

$\overline{z_1\pm z_2}=\overline{z_1}\pm\overline{z_2}$	$\overline{z_1\times z_2}=\overline{z_1}\times\overline{z_2}$
$z+\bar z=2Re(z)=2x$	
$z-\bar z=2iIm(z)=2yi$	
$z\times\bar z=x^2+y^2=\lvert z\rvert^2=r^2$	
$\frac{z}{\bar z}=\left(\frac{x^2-y^2}{x^2+y^2}\right)+i\left(\frac{2xy}{x^2+y^2}\right)$	

- Rules for Arguments of Complex Numbers:

$\arg(z\times w)=\arg(z)+\arg(w)$
$\arg(z\div w)=\arg(z)-\arg(w)$

- Rules for Moduli of Complex Numbers:

$\lvert z\times w\rvert=\lvert z\rvert\times\lvert w\rvert$	$\left\lvert\frac zw\right\rvert=\frac{\lvert z\rvert}{\lvert w\rvert}$

- Simplifying Complex Numbers:

$$z^{-1}=\frac1z=\frac{1}{x+yi}\times\frac{x-yi}{x-yi}=\frac{x-yi}{x^2+y^2}=\frac{\bar z}{|z|}$$

$$\frac zw=\frac{a+bi}{c+di}=\frac{a+bi}{c+di}\times\frac{c-di}{c-di}=\frac{z\times\bar w}{|w|^2}$$

COMPLEX ROOTS

Quadratic Formula

- Finds real and complex roots of a quadratic equation in the form of $y=ax^2+bx+c$:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Finding Complex Roots Examples

(Q1a) Find all the roots of $y=x^2-4x+5$:
- Substituting $a=1,b=-4,c=5$:

$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)}=\frac{4\pm\sqrt{16-20}}{2}$
$=\frac{4\pm\sqrt{-4}}{2}=\frac{4\pm\sqrt4\sqrt{-1}}{2}=\frac{4\pm2i}{2}=2\pm i$
$\therefore$ roots of the function is $x=\mathbf{2+i,2-i}$

(Q1b) Find the linear factorisation of equation:
- Using the null factor law:

$y=x^2-4x+5=(x-(2+i))(x-(2-i))$
$y=(x-2-i)(x-2+i)$

(Q1c) Describe the nature of these roots:
Roots of the equation, $x=2\pm i$, are complex **conjugates** as $Re(x)$ stays the same and the sign of $Im(x)$ changes between ±.

COMPLEX ALGEBRA

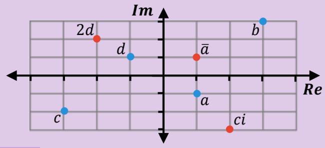
Complex Number Algebra Examples

(Q1) Find the real and imaginary components and the conjugate for the complex numbers:

z	$Re(z)$	$Im(z)$	$\bar z$
$-3+2i$	-3	2	$-3-2i$
$-\sqrt2i+5$	5	$-\sqrt2$	$\sqrt2i+5$
6	6	0	6
$-i$	0	-1	i

(Q2) Use the properties of imaginary numbers to simplify the following expressions:
(Q2a) $\sqrt{-49}=7i$ **(Q2b)** $-4\sqrt{-9}=-12i$
(Q2c) i^{87} **(Q2d)** $8i^2\sqrt{-36}$
$87\div4=21r3$ *Find remainder $=8(-1)(6i)$
$\therefore i^{87}=-i$ $=-48i$

(Q3a) Plot these points on the argand diagram:
$a=2-i,b=3+3i,c=-3-2i,d=-1+i$



(Q3b) Plot $\bar a$ and describe the transformation:
$\bar a=2+i$ which reflects a in the x-axis.
(Q3b) Plot ci and describe the transformation:
$ci=i(-3-2i)=-3i-2i^2=2-3i$ which rotates c anti-clockwise about the origin.
(Q3c) Plot $2d$ and describe the transformation:
$2d=2(-1+i)=-2+2i$ which increases the modulus of d by a scale factor of 2.

(Q4) Given the two different complex numbers $w=2-3i$ and $z=-1+4i$, find the following:
(Q4a) Simplify the expression $2w+\bar z$:
$2w+\bar z=2(2-3i)-1-4i=4-6i-1-4i$
$\therefore 2w+\bar z=3-10i$
(Q4b) Simplify the expression $w\bar z$:
$w\bar z=(2+3i)(-1-4i)=-2-8i-3i-12i^2$
$w\bar z=-2-11i-12(-1)=10-11i$
(Q4c) Simplify the expression i^2w-iz:
$i^2w-iz=(-1)(2-3i)-i(-1+4i)$
$i^2w-iz=-2+3i+i-4i^2=2+4i$
(Q4d) Simplify the expression $w\div z$:
- To simplify a complex number divided by another, multiply it by a fraction where the numerator and denominator are conjugate $\bar z$:

$\frac wz=\frac wz\times1=\frac wz\times\frac{\bar z}{\bar z}=\frac{2-3i}{-1+4i}\times\frac{-1-4i}{-1-4i}$
$\frac wz=\frac{(2-3i)(-1-4i)}{(-1+4i)(-1-4i)}=\frac{-2-8i+3i+12i^2}{1+4i-4i-16i^2}$
$\frac wz=\frac{-14-5i}{17+0i}=\frac{-14-5i}{17}=-\frac{14}{17}-\frac{5}{17}i$

(Q4e) Simplify the expression $y=\frac{w}{1+i}+\frac{z}{1-i}$ and find the value of k that satisfies $Re(y)=kIm(y)$:
$y=\frac{w}{1+i}+\frac{z}{1-i}=\frac{2-3i}{1+i}+\frac{-1+4i}{1-i}$ *Cross-multiply fractions
$y=\frac{(1-i)(2-3i)+(-1+4i)(1+i)}{(1+i)(1-i)}$
$y=\frac{(2-3i-2i+3i^2)+(-1-i+4i+4i^2)}{1+i-i-i^2}$
$y=\frac{1-2i+3i^2+4i^2}{1+1}=\frac{-6-2i}{2}=-3-i$
Hence, $Re(y)=-3$ and $Im(y)=-1$
$Re(y)=kIm(y)\to-3=-1k\to k=3$

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